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ON QUANTUM ELECTRODYNAMICS

BY

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Introduction.

In the formulation of quantum electrodynamics as introduced by FERMI, the state of a system has to obey a supplementary condition and the Maxwell equations are not valid as operator equations, but only as derived supplementary conditions. Several authors have pointed out the inconsistencies[1]), [2]), [3]) which arise from the fact that the Hilbert space introduced to characterize the state of the system does not contain elements satisfying the supplementary condition. On the other hand, a considerable number of contributions have been made in recent years to elucidate the way in which the longitudinal field variables appear in the FERMI theory and its connection to the Coulomb interaction[1])-[7]), and progress has been made in many respects in the understanding of the problem. A rather radical change in the interpretation of the scheme by means of the indefinite metric of Dirac has also been proposed[8]).

We want to approach the problem of the formulation of quantum electrodynamics in this paper from a different point of view. By means of a new quantization method, NOVOBÁTZKY[9]) has given a canonical formulation of quantum electrodynamics with a separated treatment of the Coulomb interaction, avoiding completely the appearance of the supplementary condition. In a second paper[10]), he proposed, in order to include from the beginning the Coulomb interaction but to avoid the supplementary condition, to introduce only two kinds of transverse and one kind of longitudinal photon variables, instead of the four kinds of photons of the Fermi theory. The proposed covariant decomposition of the potentials which applies well in the meson case seems to lead, however, to difficulties in the electromagnetic case, owing to the singularities in the operators introduced. Therefore, in taking over the idea of Novobátzky of introducing only three kinds of photons in order to describe the electromagnetic interactions, we follow a quite different line and the form of the resulting theory will in this way also be different.

Whereas the decomposition of a potential vector into transverse and longitudinal parts is always connected with a special choice of the time axis, the difference in the physical meaning and the role played by the transverse and longitudinal photons leads to the conclusion that the distinction between the states of free transverse and longitudinal photons has to be a relativistic one. Starting from the interaction representation, one can characterize the states of transverse photons in a relativistic way by means of the 6-vector solutions of the vacuum Maxwell equations which correspond in any reference system to transverse waves only. In order to characterize longitudinal photon states we introduce then another, scalar field. The interaction with the electrons can then be described by defining potentials given by these fields and related to a given time-like direction (or to a given space-like surface). These potentials satisfy commutation relations depending on the given time-like vector.

By means of a canonical transformation, very similar to that used in the Fermi theory, one can eliminate the variables of the scalar field and obtain the wave equation of the usual reduced theory with a Hamiltonian which is the sum of the transverse interaction energy and of the Coulomb energy. Transforming the equations from the interaction representation to the Heisenberg representation, we obtain potentials whose equations depend explicitly on the special choice of the gauge and which do not satisfy the Lorentz condition. The field strengths formed by means of these potentials do not depend, however, on the scalar photon variables and on the special gauge, and satisfy the inhomogeneous Maxwell equations. Finally, we show that in calculating the S-matrix, the commutation relations of the potentials can be replaced by the simpler ones of the Fermi theory, since the additional terms in the commutation relations do not give any contribution.

As shown by Prof. C. Møller, one can build up the theory also by starting directly from the Heisenberg picture, and introducing suitable energy-momentum expressions and the corresponding commutation rules. Some aspects of the theory become clearer in the Heisenberg picture and, in writing down the formal solution of the equations, one can also get a clearer insight into the transmission of the Coulomb interaction by means of the longitudinal waves. The detailed discussion of questions related to the Heisenberg representation will form the subject of a forthcoming paper.

Interaction representation.

(a) *Transverse photon states.*

A transverse photon state can be described by means of a 6-vector $F_{\mu\nu}^{(1)}$ satisfying the vacuum Maxwell equations

$$\partial_\mu F_{\mu\nu}^{(1)} = 0. \tag{1}$$

$$\partial_\varkappa F_{\mu\nu}^{(1)} + \partial_\nu F_{\varkappa\mu}^{(1)} + \partial_\mu F_{\nu\varkappa}^{(1)} = 0. \tag{2}$$

The equations (2) express that $F_{\mu\nu}^{(1)}$ can be derived from a 4-vector $A_\nu^{(1)}$ as

$$F_{\mu\nu}^{(1)} = \partial_\mu A_\nu^{(1)} - \partial_\nu A_\mu^{(1)}. \tag{2 a}$$

Since two of the equations (1), (2) are $\partial_1 F_{14}^{(1)} + \partial_2 F_{24}^{(1)} + \partial_3 F_{34}^{(1)} = 0$, $\partial_1 F_{23}^{(1)} + \partial_2 F_{31}^{(1)} + \partial_3 F_{12}^{(1)} = 0$, $F_{\mu\nu}^{(1)}$ represents a transverse field in any reference system. To every light vector k_μ, $k_\mu^2 = 0$, correspond two independent solutions of the equation system (1), (2), characterizing the two kinds of polarization of a plane wave. In writing equations (1), (2) in the form of a particle wave equation, one can also give a simple interpretion to the quantities related to the particle aspect of radiation theory[11]).

The quantization of the vacuum equations (1), (2) can be performed in known ways[12]). For the hermitian operators $F_{\mu\nu}^{(1)}$ giving the field strength in the interaction representation we obtain the commutation relations

$$\left.\begin{aligned} &\left[F_{\mu\nu}^{(1)}(x),\, F_{\lambda\varkappa}^{(1)}(x')\right] = \\ &i\left\{\delta_{\nu\lambda}\,\partial_\mu\,\partial_\varkappa + \delta_{\mu\varkappa}\,\partial_\nu\,\partial_\lambda - \delta_{\nu\varkappa}\,\partial_\mu\,\partial_\lambda - \delta_{\mu\lambda}\,\partial_\nu\,\partial_\varkappa\right\} D(x-x').\end{aligned}\right\} \tag{3}$$

We use units with $c = 1$, $\hbar = 1$; the sign of the invariant function $D(x-x')$ is that used by SCHWINGER[5]). These commutation relations can also be obtained, following a method of NOVOBÁTZKY[9]), by deriving $F^{(1)}_{\mu\nu}$ from two quantities Q_a, Q_b characterizing linearly polarized waves and related to canonical commutation relations. One can also introduce two scalar (invariant) functions Q_1, Q_2 related to circularly polarized waves. Some more details about the free fields will also be given in the forthcoming paper mentioned above.

In describing the states of free electrons by means of the Dirac equation, we want to introduce the interaction of the electrons with the electromagnetic field first in the interaction representation. As pointed out especially by COESTER and JAUCH[3]), the covariant aspect of the calculations in electrodynamics is fully preserved in relating the state of the system to a hyperplane σ, defined by a time-like direction n_μ, $n_\mu n_\mu = -1$, instead of introducing more general space-like surfaces. We shall accept this point of view throughout, σ meaning in the following always a plane perpendicular to n_μ. In denoting by τ a time parameter measured in the direction n_μ, the state Φ of the system of electrons and photons satisfies in the interaction representation a wave equation of the form

$$i \frac{\partial}{\partial \tau} \Phi = H_1 \Phi. \tag{4}$$

The part of H_1 corresponding to the interaction energy of the electrons with the transverse field $F^{(1)}_{\mu\nu}$ can be written in the form

$$H_1^{(1)} = -\int d\sigma' j_\mu(x') A_\mu^{(1)}(x'). \tag{5}$$

$j_\mu(x')$ is the current operator of the Dirac electrons in the interaction representation, and the potential $A_\mu^{(1)}(x')$ will be defined now by means of the field $F^{(1)}_{\mu\nu}$.

Using the notation of COESTER and JAUCH[3]), we write $\partial = n_\nu \partial_\nu$, and write ∂^{-1} for the inverse operator which, in the case when a Fourier expansion is possible, means a multiplication of each Fourier component by $(i\, n_\nu k_\nu)^{-1}$. With this notation, we define a transverse potential $A_\mu^{(1)}$ related to the time-like direction n_μ by

$$A_\nu^{(1)} = \partial^{-1} F_{\mu\nu}^{(1)} n_\mu . \tag{6}$$

One has

$$A_\nu^{(1)} n_\nu = 0 . \tag{6a}$$

$$\partial_\nu A_\nu^{(1)} = 0 . \tag{6b}$$

(6a) is the consequence of the antisymmetry of $F_{\mu\nu}^{(1)}$, (6b) follows from equation (1). The relation (2a) is fulfilled by (6), owing to equation (2). This shows also that a different choice of the time-like direction n_μ means only a different choice of the gauge of the potential $A_\nu^{(1)}$. From (1), (2a), (6b) we have also

$$\Box A_\nu^{(1)} = 0 . \tag{6c}$$

$A_\nu^{(1)}$ satisfies, according to (6) and (3), the commutation relations

$$\left[A_\mu^{(1)}(x), A_\nu^{(1)}(x')\right] = i d_{\mu\nu}^{(1)} D(x-x') \tag{7}$$

with

$$d_{\mu\nu}^{(1)} = \delta_{\mu\nu} - \partial_\mu \partial_\nu \partial^{-2} - n_\mu \partial_\nu \partial^{-1} - n_\nu \partial_\mu \partial^{-1} . \tag{7a}$$

These are the same commutation relations as those of the transverse potential related to the time-like direction n_μ of the Fermi theory, as given for instance by SCHWINGER[5]).

Writing the potential (6) in (5), and taking for the Hamiltonian of the wave equation (4) $H_1 = H_1^{(1)} + H_C$, where H_C is the expression for the Coulomb interaction energy in covariant form, the content of the theory is exactly the same as that of the usual treatments where the Coulomb energy is added separately to the interaction energy of charged particles and light waves. The formulation presented here has, however, the advantage that the states of the light quanta are described in a relativistic way by means of the 6-vector functions $F_{\mu\nu}^{(1)}$.

(b) *Scalar photon states.*

Recent calculations in quantum electrodynamics have shown that one can treat many problems more easily by dealing with the Coulomb energy on the same footing as the interaction with the light waves. Also for physical reasons one has to avoid the instantaneous aspect of the Coulomb interaction, in ascribing it to interactions transmitted by a field. In the scheme, as sketched here until now, there is, however, no place left for interactions by means of a longitudinal field. The field $F_{\mu\nu}^{(1)}$ obeying equations (1) and (2) is completely transverse.

We introduce, therefore, a new field in order to describe longitudinal interactions and choose it in such a way that it can give account of the Coulomb interaction. In the quantized theory, this interaction will correspond to the virtual emission and absorption of quanta. This field and these quanta will, however, not represent measurable quantities, but will be related only to the gauge of the potentials. This will correspond to the fact that also in the classical theory the retarded transmission of Coulomb interactions is related only to potential waves. Since the homogeneous Maxwell equations for the field strengths have only transverse solutions, the Coulomb force in the corresponding inhomogeneous equations has also an instantaneous appearance.

We introduce a 4-vector field B_μ, satisfying in the vacuum equations analogous to (1) and (2)

$$\partial_\mu B_\mu = 0. \tag{8}$$

$$\partial_\nu B_\varkappa - \partial_\varkappa B_\nu = 0. \tag{9}$$

From (9) one can write

$$B_\mu = \partial_\mu Q \tag{9a}$$

and in this way derive B_μ from a single scalar function $Q(x)$. B_μ being a 4-gradient, its space component is in every reference system a longitudinal vector. The canonical formalism of the equations (8), (9) can easily be worked out. It corresponds to the theory of a scalar meson with zero rest mass[12]).

We want to characterize the states of scalar photons by means of the functions B_μ or Q. In quantizing the theory we choose commutation relations

$$[Q(x), Q(x')] = -iD(x-x'). \tag{10}$$

We shall come back to the question of the minus sign in (10). It corresponds to negative energy quanta as in the case of the scalar photons of the Fermi theory. As remarked, we do not attribute any observable physical meaning to these quanta.

We want to introduce the interaction of the electrons with these scalar photons by adding in the wave equation (4) of the interaction representation to the transverse interaction energy (5) another term of the form

$$H_1^{(2)} = -\int_\sigma d\sigma' j_\mu(x') A_\mu^{(2)}(x'). \tag{11}$$

We define the potential $A_\mu^{(2)}$ with respect to the time-like direction n_μ as

$$A_\mu^{(2)} = -\partial^{-1} B_\mu = -\partial_\mu \partial^{-1} Q. \tag{12}$$

From (10), we have for $A_\mu^{(2)}$ the commutation relations

$$\left[A_\mu^{(2)}(x), A_\nu^{(2)}(x')\right] = i d_{\mu\nu}^{(2)} D(x - x') \tag{13}$$

$$d_{\mu\nu}^{(2)} = -\partial_\mu \partial_\nu \partial^{-2}. \tag{13a}$$

From (12) we have evidently

$$F_{\mu\nu}^{(2)} = \partial_\mu A_\nu^{(2)} - \partial_\nu A_\mu^{(2)} = 0. \tag{12a}$$

From (8) and (12)

$$-n_\mu A_\mu^{(2)} = Q \tag{12b}$$

$$\partial_\mu A_\mu^{(2)} = 0. \tag{12c}$$

From (8), (9a), and (12)

$$\Box A_\mu^{(2)} = 0. \tag{12d}$$

In introducing the potential

$$A_\mu = A_\mu^{(1)} + A_\mu^{(2)}, \tag{14}$$

we have from (6b), (6c), (12c), (12d)

$$\Box A_\mu = 0, \quad \partial_\mu A_\mu = 0, \tag{14a}$$

and from (7) and (13)

$$[A_\mu(x), A_\nu(x')] = i d_{\mu\nu} D(x - x') \tag{15}$$

$$d_{\mu\nu} = d_{\mu\nu}^{(1)} + d_{\mu\nu}^{(2)} = \delta_{\mu\nu} - 2\,\partial_\mu\,\partial_\nu\,\partial^{-2} - n_\mu\,\partial_\nu\,\partial^{-1} - n_\nu \partial_\mu \partial^{-1}. \tag{15a}$$

With (5), (11), and (14), we can write for the Hamiltonian of the wave equation (4)

$$H_1 = H_1^{(1)} + H_1^{(2)} = -\int_\sigma d\sigma'\, j_\mu(x')\, A_\mu(x'). \tag{16}$$

As we shall see, the wave equation (4), (16) together with the commutation relations (15) describes correctly the interaction between electrons and the electromagnetic field.

Considerable effort has been made in recent publications to define the vacuum state in the Fermi electrodynamics. In the present theory, the vacuum state can be simply taken as the state in which there are no electrons, no positrons, no transverse and no scalar photons present. As a consequence of this definition, the annihilation operators of single particle states give zero if applied to the vacuum state Φ_0. With respect to the photon variables this can be written in the form

$$F_{\mu\nu}^{(1)+}(x)\,\Phi_0 = 0, \qquad Q^-(x)\,\Phi_0 = 0, \tag{17}$$

where the $\pm$ sign denotes the positive and negative frequency parts of the corresponding operators. These equations are naturally independent of the time-like direction n_μ. Owing to the fictional character of the scalar quanta which are related only to the gauge of the potentials, much significance should not be attributed, however, to the second of the conditions (17).

Though we have written the equations and commutation relations in covariant notations, this does not imply in itself the relativistic invariance of the scheme. The commutation relations (7), (13), (15) depend explicitly on a time-like vector n_μ, and the Hamiltonian (16) is defined with respect to a reference system in which n_μ is the time axis. Nevertheless, the scheme is not only covariant in its notations, but relativistic also in its content. As to the form-invariance of the commutation relations, this can be seen from the following remarks*. The commutation

* The elucidation of this point is the result of discussions with Prof. C. MøLLER. Other aspects of the question will be dealt with in the referred forthcoming paper.

relations (7) follow from (3) and (6). Conversely, (3) follows from (7) and (2a). In the same way, the commutation relations (13) follow from (10), (12), and (10) follows from (13), (12b). (3) and (10) do not depend on n_μ and are independent of the reference system. The relation between the potentials $A_\mu^{(1)}, \overline{A}_\mu^{(1)}$ defined by (6) for two different time-like vectors n_μ, $\overline{n}_\mu$ can be obtained in writing in (2a) the potential $\overline{A}_\mu^{(1)}$ and substituting this expression in (6). The equations (12b) and (12) define in the same way a relation between $A_\mu^{(2)}$ and $\overline{A}_\mu^{(2)}$. Starting from these relations one can easily see that if (7) and (13) are valid for the potentials $A_\mu^{(1)}$, $A_\mu^{(2)}$ defined with respect to n_μ, the same relations are valid with $\overline{n}_\mu$ instead of n_μ for the potentials $\overline{A}_\mu^{(1)}$, $\overline{A}_\mu^{(2)}$; for (7) and (13) involve the relations (3) and (10) which are independent of n_μ, and these involve again (7) and (13) with $\overline{n}_\mu$ instead of n_μ. The commutation relations which have the same form in every reference system follow in this way from each other, and the wave equation (4), (16) has also the same form in every system.

The elimination of the scalar photon variables. Coulomb interaction.

The simplest way of showing that the effect of the introduced scalar field and of the interaction term (11) is simply the transmission of the Coulomb interaction, and that in this way the physical results of the present formulation are the same as those of other formulations of quantum electrodynamics, is to obtain by a canonical transformation the elimination of the scalar photon variables and the direct appearance of the Coulomb energy.

Let us transform the wave equation (4), (16) by means of the canonical transformation

$$\Phi(\tau) = e^{i\Sigma}\chi(\tau) \tag{18}$$

into the form

$$i\frac{\partial}{\partial\tau}\chi(\tau) = G\chi(\tau) \tag{19}$$

where

$$G = e^{-i\Sigma} H_1 e^{i\Sigma} - i e^{-i\Sigma} \frac{\partial}{\partial \tau} e^{i\Sigma} = H_1 + \dot{\Sigma} + i [H_1, \Sigma] + \frac{i}{2} [\dot{\Sigma}, \Sigma]. \quad (20)$$

The second equality in (20) is valid if $[H_1, \Sigma]$ and $[\dot{\Sigma}, \Sigma]$ commute with Σ. This is the case for

$$\Sigma = \int_\sigma d\sigma\, n_\mu j_\mu(x)\, \partial^{-1} Q(x) = -\int_\sigma d\sigma\, n_\mu j_\mu(x)\, \partial^{-1} n_\nu A_\nu^{(2)}(x) \quad (21)$$

for which the Gauss theorem gives with (12) and (11)

$$\dot{\Sigma} = \frac{\partial}{\partial \tau} \Sigma = -\int_\sigma d\sigma'\, j_\nu(x')\, \partial_\nu \partial^{-1} Q(x') = -H_1^{(2)}. \quad (22)$$

We have further, using the commutation relation (10),

$$\left.\begin{aligned} [\dot{\Sigma}, \Sigma] &= -[H_1^{(2)}, \Sigma] = -[H_1, \Sigma] = \\ &= -\int_\sigma d\sigma \int_\sigma d\sigma'\, n_\mu j_\mu(x) j_\nu(x') \left[\partial_\nu \partial^{-1} Q(x'), \partial^{-1} Q(x)\right] = \\ &= -i \int_\sigma d\sigma \int_\sigma d\sigma'\, n_\mu j_\mu(x) j_\nu(x')\, \partial_\nu \partial^{-2} D(x-x'). \end{aligned}\right\} \quad (23)$$

The four terms of (20) give with (22), (23)

$$\left.\begin{aligned} G = H_1 - H_1^{(2)} &+ \int_\sigma d\sigma \int_\sigma d\sigma'\, n_\mu j_\mu(x) j_\nu(x') \\ &\left\{-\partial_\nu \partial^{-1} + \frac{1}{2} \partial_\nu \partial^{-1}\right\} \partial^{-1} D(x-x') = H_1^{(1)} + H_C \end{aligned}\right\} \quad (24)$$

where

$$\left.\begin{aligned} H_C &= -\frac{1}{2} \int_\sigma d\sigma \int_\sigma d\sigma'\, n_\mu j_\mu(x) j_\nu(x')\, \partial_\nu \partial^{-2} D(x-x') = \\ &= \frac{1}{2} \int_\sigma d\sigma \int_\sigma d\sigma'\, n_\mu j_\mu(x)\, n_\nu j_\nu(x')\, \partial^{-1} D(x-x') \end{aligned}\right\} \quad (25)$$

is the covariant expression of the Coulomb energy. The second form of (25) is obtained by using the fact that on the hyperplane

σ one has $(\partial_\nu + n_\nu \partial)\, \partial^{-2} D(x - x') = 0$. In the special system, with n_μ as time axis, $\partial^{-1} D(x - x') \simeq \frac{1}{4\pi r}$. $H_1^{(1)}$ is the interaction energy (5) with the transverse radiation field, and the Hamiltonian of the wave equation (19) does not contain any longer the scalar field variables. We can see from (23) that, in order to get the right sign in the Coulomb energy, we had to choose in the commutation relation (10) for $Q(x)$ the sign corresponding to the time-component photons of the Fermi electrodynamics.

On the elimination of the longitudinal variables in the Fermi theory.

At this stage, it seems instructive to compare the canonical transformations proposed by different authors in order to eliminate the longitudinal field variables in the Fermi electrodynamics. In this case, the interaction Hamiltonian of the wave equation

$$i \frac{\partial}{\partial \tau} \Phi = H_1 \Phi \tag{26}$$

$$H_1 = - \int_\sigma d\sigma' \, j_\mu(x') \, A_\mu(x') \tag{26a}$$

contains the potential components with the commutation relations

$$[A_\mu(x), A_\nu(x')] = i \delta_{\mu\nu} D(x - x'). \tag{27}$$

The state Φ has to satisfy the supplementary condition

$$\left.\begin{aligned} &\Omega(x, \tau)\, \Phi(\tau) = 0 \\ \text{with}\quad &\Omega(x, \tau) = \partial_\mu A_\mu(x) + \int_{\sigma(\tau)} d\sigma' \, n_\mu j_\mu(x')\, D(x - x'). \end{aligned}\right\} \tag{28}$$

The potential components $A_\mu(x)$ obey the equations $\square A_\mu(x) = 0$.

By means of the operator $d_{\mu\nu}^{(1)}$ of (7 a) we can define a transverse potential

$$\left.\begin{aligned}\mathscr{A}_\mu(x) &= d^{(1)}_{\mu\nu} A_\nu(x) = \\ &= \left\{ \delta_{\mu\nu} - \partial_\mu \partial_\nu \partial^{-2} - n_\mu \partial_\nu \partial^{-1} - n_\nu \partial_\mu \partial^{-1} \right\} A_\nu(x)\end{aligned}\right\} \tag{29}$$

for which

$$n_\mu \mathscr{A}_\mu = 0, \quad \partial_\mu \mathscr{A}_\mu = 0. \tag{30}$$

(29) can be written in the alternative forms

$$\mathscr{A}_\mu(x) = \left\{\delta_{\mu\nu} + n_\mu n_\nu - (\partial_\mu \partial^{-1} + n_\mu)(\partial_\nu \partial^{-1} + n_\nu)\right\} A_\nu(x) \tag{29a}$$

$$\mathscr{A}_\mu(x) = \left\{\delta_{\mu\nu} - \partial_\mu \partial^{-1} n_\nu - (\partial_\mu \partial^{-1} + n_\mu)\partial_\nu \partial^{-1}\right\} A_\nu(x) \tag{29b}$$

$$\mathscr{A}_\mu(x) = \left\{\delta_{\mu\nu} - \frac{1}{2}\partial_\mu \partial^{-1}(\partial_\nu \partial^{-1} + 2n_\nu) - \frac{1}{2}(\partial_\mu \partial^{-1} + 2n_\mu)\partial_\nu \partial^{-1}\right\} A_\nu(x). \tag{29c}$$

(29a) corresponds to the decomposition of SCHWINGER[5]). The second term is a vector in the n_μ direction, the third term is in the special system the longitudinal potential given by a space-gradient. In this form of $d^{(1)}_{\mu\nu}$ one can see clearly that, for $\Box A_\mu = 0$, $d^{(1)}_{\mu\nu}$ is the projection operator of the transverse 4-vectors belonging to the time-like direction n_μ. It is defined as the difference of the unit operator and of the projection operators of the n_μ direction and of the perpendicular longitudinal direction. (29) satisfies the commutation relations (7).

(29b) is the transverse potential in the form defined and used by COESTER and JAUCH[3]). The second term is a 4-gradient, the third term depends only on $\partial_\nu A_\nu$. (29c) corresponds to the decomposition used by KOBA, TATI and TOMONAGA[4]) and by HU[6]).

One can arrive in the Fermi theory by means of different canonical transformations (18) to the direct appearance of the Coulomb interaction energy in the transformed equation (19). Such transformations are defined by

$$\Sigma = -\int_\sigma d\sigma\, n_\mu j_\mu(x)\, \partial^{-1}(\partial_\nu \partial^{-1} + n_\nu) A_\nu(x) \tag{31a}$$

$$\Sigma = -\int_\sigma d\sigma\, n_\mu j_\mu(x)\, \partial^{-1} n_\nu A_\nu(x) \tag{31b}$$

$$\Sigma = -\int_\sigma d\sigma\, n_\mu j_\mu(x)\, \partial^{-1}\left(\frac{1}{2}\partial_\nu \partial^{-1} + n_\nu\right) A_\nu(x). \tag{31c}$$

(31a) corresponds to the transformation used by SCHWINGER, (31b) is the transformation of COESTER and JAUCH, (31c) corresponds to the transformation used by KOBA, TATI and TOMONAGA and by HU.

$\dot{\Sigma}$ results in all three cases by the application of Gauss' theorem, and the commutators* in (20) by the commutation relations (27).

We obtain in this way in the three cases for the four terms of (20)

$$\left.\begin{aligned} G = H_1 + \int_\sigma d\sigma j_\mu(x)\, \partial_\mu \partial^{-1} (\partial_\nu \partial^{-1} + n_\nu)\, A_\nu(x) + \\ + \int_\sigma d\sigma \int_\sigma d\sigma'\, n_\mu j_\mu(x)\, j_\nu(x') \left\{ (\partial_\nu \partial^{-1} + n_\nu) - \frac{1}{2} \partial_\nu \partial^{-1} \right\} \partial^{-1} D(x - x') \end{aligned}\right\} \quad (32a)$$

$$\left.\begin{aligned} G = H_1 + \int_\sigma d\sigma j_\mu(x)\, \partial_\mu \partial^{-1}\, n_\nu A_\nu(x) + \\ + \int_\sigma d\sigma \int_\sigma d\sigma'\, n_\mu j_\mu(x)\, j_\nu(x') \left\{ n_\nu + \frac{1}{2} \partial_\nu \partial^{-1} \right\} \partial^{-1} D(x - x') \end{aligned}\right\} \quad (32b)$$

$$\left.\begin{aligned} G = H_1 + \int_\sigma d\sigma j_\mu(x)\, \partial_\mu \partial^{-1} \left(\frac{1}{2} \partial_\nu \partial^{-1} + n_\nu \right) A_\nu(x) + \\ + \int_\sigma d\sigma \int_\sigma d\sigma'\, n_\mu j_\mu(x)\, j_\nu(x') \left\{ \left(\frac{1}{2} \partial_\nu \partial^{-1} + n_\nu \right) + 0 \right\} \partial^{-1} D(x - x'). \end{aligned}\right\} \quad (32c)$$

We have in all three cases

$$\left.\begin{aligned} i\,[\partial_\mu A_\mu(x), \Sigma] = -\int d\sigma'\, n_\mu j_\mu(x')\, D(x - x') \\ e^{-i\Sigma}\, \Omega(x, \tau)\, e^{i\Sigma} = \partial_\mu A_\mu(x) \end{aligned}\right\} \quad (33)$$

and the supplementary condition (28) is transformed by (18) into

$$\partial_\mu A_\mu(x)\, \chi(\tau) = 0. \quad (34)$$

* The calculation of these involves only the commutability of the current components $j_\nu(x')$ with the time-like component $n_\mu j_\mu(x)$ on the surface σ. Contrary to the statement in ref. (3) and (8_3), the current operators $j_\nu(x')$ and $j_\mu(x)$ of Dirac electrons themselves in general do not commute on a space-like surface (in the point of coincidence).

From the point of view of states $\chi(\tau)$ satisfying (34), the transformations (31 a, b, c) are identical and lead to the same states $\Phi(\tau) = e^{i\Sigma}\chi(\tau)$. The first two terms of (32 a, b, c) reduce for these states to the expression of the interaction energy with the transverse potential. From the point of view of the Fermi field, however, with the commutation relations (27), which reduces only for states satisfying the supplementary condition to the Maxwell field of electrodynamics, the canonical transformations defined by (18) and (31 a, b, c) are different. All of them lead to the appearance of the last terms in (32 a, b, c) which are equal to the covariant expression (25) of the Coulomb energy. As pointed out especially by COESTER and JAUCH in the case of the transformation (31 b), the appearance of this Coulomb term is quite independent of the supplementary condition. The supplementary condition is only used to reduce the first two terms of (32 a, b, c) to the expression of the transverse interaction energy.

The origin of the Coulomb term is, however, very different in the three cases. In the case (32 a) of SCHWINGER, since we have $(\partial_\nu \partial^{-1} + n_\nu)\,\partial^{-1} D(x-x') = 0$ on σ, the transformation of the interaction energy H_1 does not contribute to the Coulomb term, and the whole expression comes from the term $-ie^{-i\Sigma}\dfrac{\partial}{\partial\tau}e^{i\Sigma}$ in (20) which corresponds in the Schrödinger representation, where Σ is time-independent, to the transformation of the Hamiltonian of the fields without interaction. In case (32 b) of COESTER and JAUCH, the transformation of H_1 gives twice the Coulomb energy which is compensated by a negative Coulomb term coming from $-ie^{-i\Sigma}\dfrac{\partial}{\partial\tau}e^{i\Sigma}$. In the case (32 c) of KOBA, TATI and TOMONAGA and of HU, the appearance of the Coulomb term is due completely to the transformation of the interaction energy H_1.

The transformation (18), (21) is closest to the transformation (31 b) of COESTER and JAUCH. In our formulation we have identically $\partial_\mu A_\mu(x) = 0$ and correspondingly changed commutation relations. The second term in the decomposition (29 b) corresponds to the longitudinal potential $A_\mu^{(2)}$. But, while (29 b) gives the decomposition of the same operator $A_\mu(x)$ for different n_μ—s, the definition (6) and (12) of $A_\mu^{(1)}(x)$, $A_\mu^{(2)}(x)$ introduces for each n_μ a different 4-vector potential $A_\mu(x)$ which is determined by the 6-vector $F_{\mu\nu}^{(1)}(x)$ and the invariant $Q(x)$. The canonical transformation (18), (21) adds to $A_\mu^{(2)}(x)$ the Coulomb potential and, as seen from (24), the transformation of H_1 leads, therefore, to twice the Coulomb energy. This has to be compensated by a negative Coulomb term coming from the energy operator of the free scalar photons. One has, therefore, to choose the minus sign in the commutation relation (10), associating in this way the transmission of the Coulomb interaction with the virtual appearance of scalar negative energy quanta.

Transition from the interaction representation to the Heisenberg representation.

In the present formulation, the introduction of the quantities in the interaction representation has some advantage owing to the relativistic distinction between light waves and scalar photons. In the Heisenberg representation, the field equations relating the interacting electromagnetic field with the currents of the electrons have to reduce to the well-known Maxwell equations. Since we have no supplementary condition in the theory, the Maxwell equations have to be valid between the operators themselves. This is to be shown now.

The operators of the interaction representation can be transformed into those of the Heisenberg representation by means of a unitary transformation U, depending on the plane σ or on τ. To the operators A_ν, j_ν of the interaction representation correspond operators $\boldsymbol{A}_\nu, \boldsymbol{j}_\nu$ of the Heisenberg representation according to the relation

$$\boldsymbol{A}_\nu = U^{-1} A_\nu U, \qquad \boldsymbol{j}_\nu = U^{-1} j_\nu U. \tag{35}$$

We write bold-face type letters for the quantities in the Heisenberg representation.

All the quantities from now on are taken on the plane σ, and for the time derivation in the perpendicular direction n_μ we can write $\partial = n_\mu \partial_\mu$. The time dependence of the operators of the interaction representation (in the n_μ direction) is given by the Hamiltonian H_0 of the system without interaction, $\partial A_\nu = i[H_0, A_\nu]$, and with the Hamiltonian $\boldsymbol{H} = \boldsymbol{H}_0 + \boldsymbol{H}_1$ we have in the Heisenberg representation

$$\left.\begin{aligned} \partial \boldsymbol{A}_\nu(x) = i[\boldsymbol{H}, \boldsymbol{A}_\nu(x)] &= U^{-1}\{\partial A_\nu(x) + i[H_1, A_\nu(x)]\}U = \\ &= U^{-1}\partial A_\nu(x)U - \int d\sigma' \boldsymbol{j}_\mu(x') d_{\mu\nu} D(x-x'). \end{aligned}\right\} \quad (36)$$

In the last form the expression (16) of the interaction energy and the commutation relations (15) have been used.

Since the derivatives in the plane σ transform in the same way as the operators (35), we have with (36) also

$$\partial_\varkappa \boldsymbol{A}_\nu(x) = U^{-1}\partial_\varkappa A_\nu(x)U + n_\varkappa \int d\sigma' \boldsymbol{j}_\mu(x') d_{\mu\nu} D(x-x'). \quad (37)$$

For the second time derivative (in the n_μ direction) we have in the Heisenberg representation*

$$\left.\begin{aligned} \partial^2 \boldsymbol{A}_\nu = i[\boldsymbol{H}, i[\boldsymbol{H}, \boldsymbol{A}_\nu]] &= U^{-1}\{\partial^2 A_\nu + i[H_1, \partial A_\nu] + \\ &+ i[H_0, i[H_1, A_\nu]] + i[H_1, i[H_1, A_\nu]]\}U. \end{aligned}\right\} \quad (38\,\text{a})$$

Using the expression (16) for H_1, the commutation relations (15), and

$$i[H_0, j_\mu] = \partial j_\mu; \qquad U^{-1}\{\partial j_\mu + i[H_1, j_\mu]\}U = \partial \boldsymbol{j}_\mu, \quad (38\,\text{b})$$

we obtain from (38 a)

$$\left.\begin{aligned} -\partial^2 \boldsymbol{A}_\nu(x) = U^{-1}(-\partial^2 A_\nu(x))U &+ \int d\sigma' \boldsymbol{j}_\mu(x') \partial d_{\mu\nu} D(x-x') + \\ &+ \int d\sigma' d_{\mu\nu} D(x-x') \partial' \boldsymbol{j}_\mu(x'). \end{aligned}\right\} \quad (38)$$

* The reasoning of formula (2, 11) of Schwinger's first paper[5]) which takes into consideration only the first two terms of the right-hand side of (38 a), though correct in the special case of the Fermi electrodynamics, leads in general, as for instance in our case, to wrong results.

This gives with $\Box A_\nu = 0$ and with $(\Box + \partial^2)\, \boldsymbol{A}_\nu(x) = U^{-1}(\Box + \partial^2)\, A_\nu(x)\, U$,

$$\Box \boldsymbol{A}_\nu(x) = \int d\sigma' \boldsymbol{j}_\mu(x')\, \partial\, d_{\mu\nu} D(x-x') + \int d\sigma'\, d_{\mu\nu} D(x-x')\, \partial' \boldsymbol{j}_\mu(x'). \tag{39}$$

On the plane σ perpendicular to n_μ, we can write, using the relations $D(x-x') = 0$, $(\partial_\mu + n_\mu \partial)\, D(x-x') = 0$, $(\partial_\mu + n_\mu \partial)\, \partial^{-2} D(x-x') = 0$, $(\partial_\mu + n_\mu \partial)(\partial_\nu + n_\nu \partial)\, \partial^{-2} D(x-x') = 0$, which follow at once in the special system, and (7 a), (13 a), (15 a),

$$\left.\begin{aligned} d_{\mu\nu} D(x-x') &= \{n_\mu(\partial_\nu + n_\nu \partial)\, \partial^{-1} + n_\nu(\partial_\mu + n_\mu \partial)\, \partial^{-1}\}\, D(x-x') = \\ &= d_{\mu\nu}^{(2)} D(x-x') \end{aligned}\right\} \tag{40 a}$$

$$d_{\mu\nu}^{(1)} D(x-x') = 0 \tag{40 b}$$

$$\partial\, d_{\mu\nu} D(x-x') = \{\delta_{\mu\nu} \partial - 2(\partial_\mu + n_\mu \partial)(\partial_\nu + n_\nu \partial)\, \partial^{-1}\}\, D(x-x') \tag{40 c}$$

$$\text{if} \quad (x_\mu - x'_\mu)\, n_\mu = 0.$$

Using (40 a), $n_\nu^2 = -1$, $n_\nu(\partial_\nu + n_\nu \partial) = 0$, we have from (37), with $\partial_\nu A_\nu(x) = 0$,

$$\partial_\nu \boldsymbol{A}_\nu = -\int d\sigma' \boldsymbol{j}_\mu(x')\, (\partial_\mu + n_\mu \partial)\, \partial^{-1} D(x-x'). \tag{41}$$

In the Heisenberg representation $\boldsymbol{A}_\nu$ does not satisfy the Lorentz condition, but $\partial_\nu \boldsymbol{A}_\nu$ has according to (41) a value which depends on the currents and on the direction n_ν with respect to which the gauge of the potential was chosen. In the special system, with $n_\nu = (0, 0, 0, i)$, (41) has the form

$$\partial_\nu \boldsymbol{A}_\nu = -\int d^3x' \boldsymbol{j}_k(x')\, \partial_k \frac{1}{4\pi r} = -\int d^3x' \frac{1}{4\pi r}\, \partial'_k \boldsymbol{j}_k(x') = \Delta^{-1} \partial_k \boldsymbol{j}_k. \tag{41 a}$$

The inverse Δ^{-1} of the Laplacian Δ is defined by the last equality. With a similar notation, we can write (41) after partial integration, in the form

$$\left.\begin{aligned}\partial_\nu \boldsymbol{A}_\nu &= -\int d\sigma' (\partial'_\mu + n_\mu \partial') \boldsymbol{j}_\mu (x') \partial^{-1} D(x-x') = \\ &= (\Box + \partial^2)^{-1} (\partial_\mu + n_\mu \partial) \boldsymbol{j}_\mu ,\end{aligned}\right\} \tag{41 b}$$

or with $\partial_\mu \boldsymbol{j}_\mu = 0$,

$$\partial_\nu \boldsymbol{A}_\nu = \partial (\Box + \partial^2)^{-1} n_\mu \boldsymbol{j}_\mu . \tag{41 c}$$

In calculating the two terms at the right-hand side of (39), we obtain with (40 c) and (41), $\partial D(x-x') = -\delta(x-x')$ on σ,

$$\left.\begin{aligned}&\int d\sigma' \boldsymbol{j}_\mu (x') \partial d_{\mu\nu} D(x-x') = \\ = -\boldsymbol{j}_\nu(x) - 2 (\partial_\nu + n_\nu \partial) &\int d\sigma' \boldsymbol{j}_\mu (x') (\partial_\mu + n_\mu \partial) \partial^{-1} D(x-x') = \\ &= -\boldsymbol{j}_\nu (x) + 2 (\partial_\nu + n_\nu \partial) (\partial_\varkappa \boldsymbol{A}_\varkappa (x))\end{aligned}\right\} \tag{42 a}$$

with (40 a), (41), (41 b), (41 c)

$$\left.\begin{aligned}&\int d\sigma' d_{\mu\nu} D(x-x') \partial' \boldsymbol{j}_\mu (x') = \\ &= (\partial_\nu + n_\nu \partial) \int d\sigma' \partial' n_\mu \boldsymbol{j}_\mu (x') \cdot \partial^{-1} D(x-x') + \\ &+ n_\nu \int d\sigma' \partial' \boldsymbol{j}_\mu (x') (\partial_\mu + n_\mu \partial) \partial^{-1} D(x-x') = \\ &= -(\partial_\nu + n_\nu \partial) (\partial_\varkappa \boldsymbol{A}_\varkappa (x)) - n_\nu \partial (\partial_\varkappa \boldsymbol{A}_\varkappa (x)) .\end{aligned}\right\} \tag{42b}$$

From (39), (42 a, b)

$$\Box \boldsymbol{A}_\nu - \partial_\nu (\partial_\varkappa \boldsymbol{A}_\varkappa) = -\boldsymbol{j}_\nu \tag{43}$$

which gives for the field strengths defined by

$$\boldsymbol{F}_{\mu\nu} = \partial_\mu \boldsymbol{A}_\nu - \partial_\nu \boldsymbol{A}_\mu \tag{43 a}$$

the Maxwell equations

$$\partial_\mu \boldsymbol{F}_{\mu\nu} = -\boldsymbol{j}_\nu . \tag{43 b}$$

The potentials are related to a gauge depending on n_μ, but the field strengths satisfy the Maxwell equations (43b) which are independent of n_μ.

From equation (37), with $\boldsymbol{A}_\nu^{(1)}$, $d_{\mu\nu}^{(1)}$ instead of $\boldsymbol{A}_\nu$, $d_{\mu\nu}$, one has, according to (40b),

$$\partial_\nu \boldsymbol{A}_\nu^{(1)} = 0, \qquad \partial_\nu \boldsymbol{A}_\nu^{(2)} = \partial_\nu \boldsymbol{A}_\nu. \tag{44}$$

(39) gives, if written with $\boldsymbol{A}_\nu^{(1)}$, $d_{\mu\nu}^{(1)}$ or $\boldsymbol{A}_\nu^{(2)}$, $d_{\mu\nu}^{(2)}$, equations of the form (43) with the covariant expression of the transverse or longitudinal current with respect to the vector n_μ on the right-hand side. The equation for $\boldsymbol{A}_\nu^{(2)}$ can be written with (44), (41c) in the form

$$\Box \boldsymbol{A}_\nu^{(2)} = \Box n_\nu (\Box + \partial^2)^{-1} n_\varkappa \boldsymbol{j}_\varkappa \tag{45}$$

which shows that $\boldsymbol{A}_\nu^{(2)}$ differs from the Coulomb potential $\boldsymbol{V}_\nu$ related to n_ν

$$\left.\begin{aligned} \boldsymbol{V}_\nu &= n_\nu (\Box + \partial^2)^{-1} n_\varkappa \boldsymbol{j}_\varkappa \\ \partial_\nu \boldsymbol{V}_\nu &= \partial_\nu \boldsymbol{A}_\nu^{(2)} = \partial_\nu \boldsymbol{A}_\nu \end{aligned}\right\} \tag{45a}$$

only by a solution of the homogeneous equation $\Box \boldsymbol{A}_\nu^{(2)} = 0$. The corresponding inhomogeneous equations for the quantities $\boldsymbol{Q}$ and $\boldsymbol{B}_\mu$ depend also on n_μ.

As to the field $\boldsymbol{F}_{\mu\nu}$, in writing

$$\left.\begin{aligned} \boldsymbol{F}_{\mu\nu} = \boldsymbol{F}_{\mu\nu}^{(1)} + \boldsymbol{F}_{\mu\nu}^{(2)}, \qquad \boldsymbol{F}_{\mu\nu}^{(1)} &= \partial_\mu \boldsymbol{A}_\nu^{(1)} - \partial_\nu \boldsymbol{A}_\mu^{(1)}, \\ \boldsymbol{F}_{\mu\nu}^{(2)} &= \partial_\mu \boldsymbol{A}_\nu^{(2)} - \partial_\nu \boldsymbol{A}_\mu^{(2)} \end{aligned}\right\} \tag{46}$$

we have with (40b) and the equations corresponding to (37)

$$\boldsymbol{F}_{\mu\nu}^{(1)} = U^{-1} F_{\mu\nu}^{(1)} U. \tag{46a}$$

According to (12a) $F_{\mu\nu}^{(2)} = 0$, but with (46), (37) we have, in stating first that with (40a), (41), (41c),

$$\int d\sigma' \boldsymbol{j}_\varkappa(x')\, d^{(2)}_{\varkappa\nu}\, D(x-x') = (\partial_\nu + n_\nu \partial) \int d\sigma' n_\varkappa \boldsymbol{j}_\varkappa(x')\, \partial^{-1} D(x-x') +$$

$$+ n_\nu \int d\sigma' \boldsymbol{j}_\varkappa(x')(\partial_\varkappa + n_\varkappa \partial)\partial^{-1} D(x-x') = -(\partial_\nu + 2 n_\nu \partial)(\Box + \partial^2)^{-1} n_\varkappa \boldsymbol{j}_\varkappa,$$

$$\left.\begin{aligned} \boldsymbol{F}^{(2)}_{\mu\nu} &= n_\mu \int d\sigma' \boldsymbol{j}_\varkappa(x')\, d^{(2)}_{\varkappa\nu} D(x-x') - n_\nu \int d\sigma' \boldsymbol{j}_\varkappa(x')\, d^{(2)}_{\varkappa\mu} D(x-x') = \\ &= (\partial_\mu n_\nu - \partial_\nu n_\mu)(\Box + \partial^2)^{-1} n_\varkappa \boldsymbol{j}_\varkappa = \partial_\mu \boldsymbol{V}_\nu - \partial_\nu \boldsymbol{V}_\mu . \end{aligned}\right\} \quad (46\,\mathrm{b})$$

(46b) is the covariant expression for the Coulomb force, and we can see explicitly that the field $\boldsymbol{F}_{\mu\nu}$ does not depend on the scalar field variables which are related only to the gauge of the potential.

On the calculation of the S-matrix.

In the reduced form in which the wave equation is

$$i \frac{\partial}{\partial \tau} \chi(\tau) = G \chi(\tau); \qquad G = H_1^{(1)} + H_C \qquad (19\,\mathrm{a})$$

with the expressions (5) and (25) of the transverse interaction energy $H_1^{(1)}$ and the Coulomb energy H_C, the present formulation of the theory is identical with that obtained by eliminating the longitudinal variables of the Fermi electrodynamics. In calculating the S-matrix in the reduced theory, we obtain the same result in both cases.

In the Fermi electrodynamics, however, the calculation of the S-matrix is much simpler in the unseparated form, and the common treatment of analogous terms is the chief advantage in comparison with the reduced theory. We want to show now that the simple rules of calculation of the unspareated Fermi theory follow also directly from the unseparated treatment of the present formulation. In the calculation of the S-matrix the commutation rules (15) of the present formulation can be replaced by the simpler rules (27) of the Fermi electrodynamics, the additional terms in (15) giving no contribution.

Writing the solution $\chi(\tau)$ of equation (19a) corresponding to an initial solution $\chi(\tau_0)$ in the form

$$\chi(\tau) = U_{\tau,\tau_0}\chi(\tau_0), \tag{47}$$

the unitary transformation U_{τ,τ_0} can be expanded according to perturbation theory in the form

$$U_{\tau,\tau_0} = 1 + U^{(1)}_{\tau,\tau_0} + \cdots + U^{(k)}_{\tau,\tau_0} + \cdots \tag{47a}$$

$U^{(k)}_{\tau,\tau_0}$ contains in the integrand k factors $G(\tau_i)$. In writing $G(\tau_i) = H^{(1)}_1(\tau_i) + H_C(\tau_i)$, we obtain a number of terms which can be classified according to the number of factors $H_C(\tau_i)$. The terms with a single factor $H_C(\tau_i)$ give a sum

$$\left.\begin{aligned} &(-i)^k \sum_{i=1}^{k} \int_{\tau_0}^{\tau} d\tau_1 H^{(1)}_1(\tau_1) \cdots \int_{\tau_0}^{\tau_{i-2}} d\tau_{i-1} H^{(1)}_1(\tau_{i-1}) \\ &\int_{\tau_0}^{\tau_{i-1}} d\tau' H_C(\tau') \int_{\tau_0}^{\tau'} d\tau_{i+1} H^{(1)}_1(\tau_{i+1}) \cdots \int_{\tau_0}^{\tau_{k-1}} d\tau_k H^{(1)}_1(\tau_k). \end{aligned}\right\} \tag{48}$$

Similarly, we obtain terms with more than one factor $H_C(\tau_i)$, and also a term with only transverse energy factors.

Following Hu[6]), but simplifying somewhat the argument which does not depend on the special reintroduction of the longitudinal field variables, we want first to show how in the case of $\tau_0 \to -\infty$, $\tau \to +\infty$ of the S-matrix, the sum of (48) and of the corresponding part of the term

$$(-i)^{k+1} \int_{\tau_0}^{\tau} d\tau_1 H^{(1)}_1(\tau_1) \int_{\tau_0}^{\tau_1} d\tau_2 H^{(1)}_1(\tau_2) \cdots \int_{\tau_0}^{\tau_k} d\tau_{k+1} H^{(1)}_1(\tau_{k+1}) \tag{49}$$

of $U^{(k+1)}_{\tau,\tau_0}$ can be brought into a form in which the identity with the analogous terms obtained by the simpler rules of the unseparated treatment of the Fermi electrodynamics becomes manifest. Analogous considerations hold for the case of the terms containing more than one Coulomb energy factor. The same argument will then lead at once to the mentioned simplification of the calculation rules in the unseparated treatment of the present formulation with the commutation relations (15).

Since $(\partial_\mu \partial^{-1} + n_\mu)\,\partial^{-1} D\,(x - x') = 0$ on the plane σ_τ, we can write the Coulomb energy (25) with[5] $D\,(x' - x'') = D^+\,(x' - x'') + D^-\,(x' - x'')$ in the form

$$\left. \begin{aligned} H_C(\tau) = \int_{\sigma_\tau} d\sigma' j_\mu(x') \int_{\sigma_\tau} d\sigma'' j_\nu(x'')\, n_\nu \left(\frac{1}{2}\,\partial_\mu \partial^{-1} + n_\mu\right) \partial'^{-1} \\ \left\{D^+\,(x' - x'') + D^-\,(x' - x'')\right\}. \end{aligned} \right\} \tag{50}$$

In introducing this form of $H_C(\tau)$ into (48), we want to transform the expression, by successive application of Gauss' theorem, in pushing the second surface integral in (50) with $D^+\,(x' - x'')$ at the right, with $D^-\,(x' - x'')$ at the left of the terms.

To obtain the necessary formulas, let us write for an arbitrary function $G\,(x, \tau)$

$$g\,(x, \tau) = \int_{\tau_0}^{\tau} d\bar{\tau}\, G\,(x, \bar{\tau}). \tag{51}$$

With the notation $g\,(x, \tau) = g\,(x/\tau)$ for x on σ_τ, we have

$$\partial g\,(x/\tau) = \frac{\partial}{\partial \tau} g\,(x/\tau) = \int_{\tau_0}^{\tau} d\bar{\tau}\, \partial G\,(x, \bar{\tau}) + G\,(x, \tau) \tag{52a}$$

$$\left. \begin{aligned} -\partial_\nu g\,(x/\tau) &= -(\partial_\nu + n_\nu \partial)\, g\,(x/\tau) + n_\nu \partial\, g\,(x/\tau) = \\ = -\int_{\tau_0}^{\tau} d\bar{\tau}\,(\partial_\nu + n_\nu \partial)\, G\,(x, \bar{\tau}) &+ \int_{\tau_0}^{\tau} d\bar{\tau}\, n_\nu \partial\, G\,(x, \bar{\tau}) + n_\nu G\,(x, \tau) = \\ &= -\int_{\tau_0}^{\tau} d\bar{\tau}\, \partial_\nu G\,(x, \bar{\tau}) + n_\nu G\,(x, \tau). \end{aligned} \right\} \tag{52b}$$

Since from (51), $g\,(x, \tau_0) = 0$, we have from the Gauss theorem and from (52b)

$$\left.\begin{aligned}&\int_{\sigma_{\tau'}} d\sigma'' j_\nu(x'')\, n_\nu g(x'',\tau') = -\int_{\tau_0}^{\tau'} d\tau'' \int_{\sigma_{\tau''}} d\sigma'' j_\nu(x'')\, \partial_\nu'' g(x''/\tau'') = \\ &= -\int_{\tau_0}^{\tau'} d\tau'' \int_{\sigma_{\tau''}} d\sigma'' j_\nu(x'') \int_{\tau_0}^{\tau''} d\bar\tau\, \partial_\nu'' G(x'',\bar\tau) + \int_{\tau_0}^{\tau'} d\tau'' \int_{\sigma_{\tau''}} d\sigma'' j_\nu(x'')\, n_\nu G(x'',\tau'').\end{aligned}\right\} \tag{53}$$

We apply now this formula to different expressions $G(x, \tau)$. With

$$G(x'', \tau_{i+1}) = \left(\frac{1}{2}\partial_\mu \partial^{-1} + n_\mu\right) \partial'^{-1} D^+(x' - x'')\, H_1^{(1)}(\tau_{i+1}) \cdots \tag{54a}$$

where the dots ... mean some other factors depending on τ_{i+1}, and with the notation $\int_{\tau_0}^{\tau'} d\tau'' \int_{\sigma_{\tau''}} d\sigma'' = \int_{\tau_0}^{\tau'} d^4 x''$, we obtain from (53), (51)

$$\left.\begin{aligned}&\int_{\sigma_{\tau'}} d\sigma'' j_\nu(x'')\, n_\nu \left(\frac{1}{2}\partial_\mu \partial^{-1} + n_\mu\right) \partial'^{-1} D^+(x' - x'') \int_{\tau_0}^{\tau'} d\tau_{i+1} H_1^{(1)}(\tau_{i+1}) \cdots = \\ &= \int_{\tau_0}^{\tau'} d^4 x'' j_\nu(x'') \int_{\tau_0}^{\tau''} d\tau_{i+1} H_1^{(1)}(\tau_{i+1}) \cdots \partial_\nu \left(\frac{1}{2}\partial_\mu \partial^{-1} + n_\mu\right) \partial^{-1} D^+(x' - x'') + \\ &+ \int_{\tau_0}^{\tau'} d\tau_{i+1} H_1^{(1)}(\tau_{i+1}) \int_{\sigma_{\tau_{i+1}}} d\sigma'' j_\nu(x'')\, n_\nu \left(\frac{1}{2}\partial_\mu \partial^{-1} + n_\mu\right) \partial'^{-1} D^+(x' - x'') \cdots.\end{aligned}\right\} \tag{54}$$

The change of sign of the first term comes from $\partial_\nu'' D^+(x' - x'') = -\partial_\nu' D^+(x' - x'')$. In the case of an even number of differential operator factors acting on the same variable we can omit the primes.

With

$$G(x'', \tau') = \int_{\sigma_{\tau'}} d\sigma' j_\mu(x') \left(\frac{1}{2}\partial_\mu \partial^{-1} + n_\mu\right) \partial'^{-1} D^-(x' - x'') \tag{55a}$$

(53), (51) gives

$$\left.\begin{aligned}&\int_{\tau_0}^{\tau_{i-1}} d\tau' \int_{\sigma_{\tau'}} d\sigma'' j_\nu(x'')\, n_\nu \int_{\sigma_{\tau'}} d\sigma' j_\mu(x') \left(\frac{1}{2}\partial_\mu \partial^{-1} + n_\mu\right) \partial'^{-1} D^-(x'-x'') = \\ &= -\int_{\tau_0}^{\tau_{i-1}} d^4 x'' j_\nu(x'') \int_{\tau_0}^{\tau''} d^4 x' j_\mu(x')\, \partial_\nu \left(\frac{1}{2}\partial_\mu \partial^{-1} + n_\mu\right) \partial^{-1} D^-(x'-x'') \cdots + \\ &+ \int_{\sigma_{\tau_{i-1}}} d\sigma'' j_\nu(x'')\, n_\nu \int_{\tau_0}^{\tau_{i-1}} d^4 x' j_\mu(x') \left(\frac{1}{2}\partial_\mu \partial^{-1} + n_\mu\right) \partial'^{-1} D^-(x'-x'') \cdots .\end{aligned}\right\} \quad (55)$$

With

$$G(x'', \tau_{i-1}) = H_1^{(1)}(\tau_{i-1}) \cdots \left(\frac{1}{2}\partial_\mu \partial^{-1} + n_\mu\right) \partial'^{-1} D^-(x'-x'') \cdots \quad (56\,\mathrm{a})$$

we obtain from (53), (51)

$$\left.\begin{aligned}&\int_{\tau_0}^{\tau_{i-2}} d\tau_{i-1} H_1^{(1)}(\tau_{i-1}) \int_{\sigma_{\tau_{i-1}}} d\sigma'' j_\nu(x'')\, n_\nu \cdots \left(\frac{1}{2}\partial_\mu \partial^{-1} + n_\mu\right) \partial'^{-1} D^-(x'-x'') \cdots = \\ &= -\int_{\tau_0}^{\tau_{i-2}} d^4 x'' j_\nu(x'') \int_{\tau_0}^{\tau''} d\tau_{i-1} H_1^{(1)}(\tau_{i-1}) \cdots \partial_\nu \left(\frac{1}{2}\partial_\mu \partial^{-1} + n_\mu\right) \partial^{-1} D^-(x'-x'') \cdots + \\ &+ \int_{\sigma_{\tau_{i-2}}} d\sigma'' j_\nu(x'')\, n_\nu \int_{\tau_0}^{\tau_{i-2}} d\tau_{i-1} H_1^{(1)}(\tau_{i-1}) \cdots \left(\frac{1}{2}\partial_\mu \partial^{-1} + n_\mu\right) \partial'^{-1} D^-(x'-x'') \cdots .\end{aligned}\right\} \quad (56)$$

The relation (54) can be used repeatedly in order to push the surface integral with $D^+(x'-x'')$ in (48) with (50) to the right. Every application of (54) gives a new term and, finally, we may use the relation

$$\left.\begin{aligned}&\int_{\sigma_{\tau_k}} d\sigma'' j_\nu(x'')\, n_\nu \left(\frac{1}{2}\partial_\mu \partial^{-1} + n_\mu\right) \partial'^{-1} D^+(x'-x'') = \\ &= \int_{\tau_0}^{\tau_k} d^4 x'' j_\nu(x'')\, \partial_\nu \left(\frac{1}{2}\partial_\mu \partial^{-1} + n_\mu\right) \partial^{-1} D^+(x'-x'') + \\ &+ \int_{\sigma_{\tau_0}} d\sigma'' j_\nu(x'')\, n_\nu \left(\frac{1}{2}\partial_\mu \partial^{-1} + n_\mu\right) \partial'^{-1} D^+(x'-x'').\end{aligned}\right\} \quad (57)$$

We obtain in this way for the corresponding part of (48)

$$\left.\begin{aligned}(-i)^k \sum_{i=1}^{k}{}' \sum_{j>i}{}' \int_{\tau_0}^{\tau} d\tau_1 H_1^{(1)}(\tau_1) \cdots \int_{\tau_0}^{\tau_{i-1}} d^4x' j_\mu(x') \cdots \int_{\tau_0}^{\tau_{j-1}} d^4x'' j_\nu(x'') \cdots \\ \int_{\tau_0}^{\tau_{k-1}} d\tau_k H_1^{(1)}(\tau_k)\, \partial_\nu \left(\frac{1}{2}\partial_\mu \partial^{-1} + n_\mu\right) \partial^{-1} D^+(x'-x'')\end{aligned}\right\} \quad (58)$$

and from the last term, according to (57),

$$\left.\begin{aligned}(-i)^k \sum_{i=1}^{k}{}' \int_{\tau_0}^{\tau} d\tau_1 H_1^{(1)}(\tau_1) \cdots \int_{\tau_0}^{\tau_{i-1}} d^4x' j_\mu(x') \cdots \\ \int_{\tau_0}^{\tau_{k-1}} d\tau_k H_1^{(1)}(\tau_k) \int_{\sigma_{\tau_0}} d\sigma'' j_\nu(x'')\, n_\nu \left(\frac{1}{2}\partial_\mu \partial^{-1} + n_\mu\right) \partial^{-1} D^+(x'-x'').\end{aligned}\right\} \quad (58a)$$

In a similar way, for the part of H_C in (48), (50), containing $D^-(x'-x'')$, we can apply first the relation (55) and then repeatedly (56) in order to push the corresponding surface integral to the left. Changing afterwards the notation according to $x' \rightleftarrows x''$, $\mu \rightleftarrows \nu$, $i \rightleftarrows j$, and using $-D^-(x''-x') = D^+(x'-x'')$, we obtain for this second part of (48)

$$\left.\begin{aligned}-(-i)^k \sum_{i=1}^{k}{}' \sum_{j<i}{}' \int_{\tau_0}^{\tau} d\tau_1 H_1^{(1)}(\tau_1) \cdots \int_{\tau_0}^{\tau_{j-1}} d^4x'' j_\nu(x'') \cdots \int_{\tau_0}^{\tau_{i-1}} d^4x' j_\mu(x') \cdots \\ \int_{\tau_0}^{\tau_{k-1}} d\tau_k H_1^{(1)}(\tau_k) \cdot \partial_\nu \left(\frac{1}{2}\partial_\mu \partial^{-1} + n_\mu\right) \partial^{-1} D^-(x'-x'') = \\ = (-i)^k \sum_{i=1}^{k}{}' \sum_{j>i}{}' \cdots \int_{\tau_0}^{\tau_{i-1}} d^4x' j_\mu(x') \cdots \int_{\tau_0}^{\tau_{j-1}} d^4x'' j_\nu(x'') \cdots \\ \partial_\mu \left(\frac{1}{2}\partial_\nu \partial^{-1} + n_\nu\right) \partial^{-1} D^+(x'-x'')\end{aligned}\right\} \quad (59)$$

and from the terms containing a surface integral at the left

$$\left.\begin{aligned}&(-i)^k \sum_{i=1}^{k} \int_{\sigma_\tau} d\sigma'' j_\nu(x'')\, n_\nu \int_{\tau_0}^{\tau} d\tau_1\, H_1^{(1)}(\tau_1) \cdots \int_{\tau_0}^{\tau_{i-1}} d^4x'\, j_\mu(x') \cdots \\ &\int_{\tau_0}^{\tau_{k-1}} d\tau_k\, H_1^{(1)}(\tau_k)\left(\frac{1}{2}\partial_\mu \partial^{-1} + n_\mu\right)\partial'^{-1} D^-(x'-x'')\end{aligned}\right\} \quad (59\mathrm{a})$$

Since we have for any finite x',

$$\lim_{\tau \to \mp \infty} \int_{\sigma_\tau} d\sigma'' j_\nu(x'')\, n_\nu \left(\frac{1}{2}\partial_\mu \partial^{-1} + n_\mu\right)\partial'^{-1} D^\pm(x'-x'') = 0, \quad (60)$$

the terms (58a), (59a) do not contribute to the S-matrix, even if we do not suppose an adiabatic switching on and switching out of the interactions. The contribution of (48) to the S-matrix is, in this way, the limit of the sum of (58) and (59) for $\tau_0 \to -\infty$, $\tau \to +\infty$,

$$\left.\begin{aligned}&(-i)^k \sum_{i=1}^{k} \sum_{j>i} \cdots \int_{\tau_0}^{\tau_{i-1}} d^4x'\, j_\mu(x') \cdots \int_{\tau_0}^{\tau_{j-1}} d^4x''\, j_\nu(x'') \cdots \\ &\cdots (\partial_\mu \partial_\nu \partial^{-2} + \partial_\mu n_\nu \partial^{-1} + \partial_\nu n_\mu \partial^{-1})\, D^+(x'-x'') \\ &\tau_0 \to -\infty, \quad \tau \to +\infty.\end{aligned}\right\} \quad (61)$$

We marked only by dots the transverse energy factors.

(49) yields an analogous term with

$$d_{\mu\nu}^{(1)} D^+(x'-x'') = (\delta_{\mu\nu} - \partial_\mu \partial_\nu \partial^{-2} - \partial_\mu n_\nu \partial^{-1} - \partial_\nu n_\mu \partial^{-1})\, D^+(x'-x'')$$

in the integrand. This cancels with (61) to the simpler term containing the factor $\delta_{\mu\nu} D^+(x'-x'')$ which corresponds to the commutation relations (27) of the Fermi electrodynamics and can be obtained directly from the unseparated treatment of this electrodynamics. This gives the result of Hu.

We return now to the question of the unseparated treatment of the theory with the commutation relations (15). Since $(\partial_\mu \partial^{-1} + n_\mu)\, \partial'^{-1} D(x'-x'') = 0$ on the plane σ_τ, we have

$$\left.\begin{aligned}&\int_{\sigma_{\tau'}} d\sigma' j_\mu(x') \int_{\sigma_{\tau'}} d\sigma'' j_\nu(x'')\, n_\nu (\partial_\mu \partial^{-1} + n_\mu)\, \partial'^{-1}\\ &\qquad\{D^+(x'-x'') + D^-(x'-x'')\} = 0.\end{aligned}\right\}\quad(62)$$

Writing, instead of (50), this expression at the place of $H_C(\tau')$ in (48), the resulting expression also vanishes. Using the relations (54), (55), (56), (57) with $(\partial_\mu \partial^{-1} + n_\mu)$ instead of $\left(\frac{1}{2}\partial_\mu \partial^{-1} + n_\mu\right)$ we obtain analogous expressions to (58), (58a), (59), (59a) and also to (60). With a result analogous to (61) we obtain in this way

$$\left.\begin{aligned}&(-i)^k \sum_{i=1}^{k}{}' \sum_{j>i}{}' \cdots \int_{\tau_0}^{\tau_{i-1}} d^4 x' j_\mu(x') \cdots \int_{\tau_0}^{\tau_{j-1}} d^4 x'' j_\nu(x'') \cdots\\ &(2\,\partial_\mu \partial_\nu \partial^{-2} + \partial_\mu n_\nu \partial^{-1} + \partial_\nu n_\mu \partial^{-1})\, D^+(x'-x'') = 0\\ &\qquad\text{for } \tau_0 \to -\infty,\ \tau \to +\infty.\end{aligned}\right\}\quad(63)$$

(63) shows that terms of the type (61), (63) of the S-matrix, calculated in the unseparated treatment with the commutation relations (15), and containing the factor

$$d_{\mu\nu} D^+(x'-x'') = (\delta_{\mu\nu} - 2\,\partial_\mu \partial_\nu \partial^{-2} - \partial_\mu n_\nu \partial^{-1} - \partial_\nu n_\mu \partial^{-1})\, D^+(x'-x'')$$

can be replaced by the simpler terms containing the factor $\delta_{\mu\nu} D^+(x'-x'')$. The contribution of the other terms vanishes. (It is understood, that the ordered products of the unseparated treatment correspond, as in the Fermi electrodynamics, to the arrangement where the negative frequency parts of the potentials stand to the left of the positive frequency parts, both for $A_\mu^{(1)}$ and $A_\mu^{(2)}$). The result is the same as if we had used the commutation rules (27) of the Fermi electrodynamics.

The combination of the results (61) and (63) shows further that the contribution (61) of (48) to the S-matrix can also be replaced by the simpler expression

$$\left.\begin{aligned} & -(-i)^k \sum_{i=1}^{k} \sum_{j>i} \cdots \int_{\tau_0}^{\tau_{i-1}} d^4 x' j_\mu(x') \cdots \\ & \int_{\tau_0}^{\tau_{j-1}} d^4 x'' j_\nu(x'') \cdots \partial_\mu \partial_\nu \partial^{-2} D^+(x'-x'') \\ & \qquad \tau_0 \to -\infty, \ \tau \to +\infty . \end{aligned}\right\} \quad (64)$$

(64) contains the factor $d_{\mu\nu}^{(2)} D^+ (x'-x'') = -\partial_\mu \partial_\nu \partial^{-2} D^+ (x'-x'')$ which corresponds to an interaction of the electrons through the field $A_\mu^{(2)} (x)$. (64) can be obtained also directly in writing the first form (25) of the Coulomb energy in (48), making the decomposition $D(x'-x'') = D^+ (x'-x'') + D^- (x'-x'')$ and using the argument corresponding to (54)—(61) with $-\frac{1}{2}\partial_\mu \partial^{-1}$ instead of $\left(\frac{1}{2}\partial_\mu \partial^{-1} + n_\mu\right)$.

Using the same arguments, but starting from the second form of H_C in (25), the resulting contribution to the S-matrix obtains the form

$$\left.\begin{aligned} & (-i)^k \sum_{i=1}^{k} \sum_{j>i} \cdots \int_{\tau_0}^{\tau_{i-1}} d^4 x' j_\mu(x') \cdots \int_{\tau_0}^{\tau_{j-1}} d^4 x'' j_\nu(x'') \cdots \\ & \frac{1}{2}(\partial_\mu n_\nu \partial^{-1} + \partial_\nu n_\mu \partial^{-1}) D^+(x'-x'') \\ & \qquad \tau_0 \to -\infty, \ \tau \to +\infty . \end{aligned}\right\} \quad (65)$$

This can be obtained also by subtracting half of the vanishing expression (63) from (61). (61), (64), (65) give alternative forms of the contribution to the S-matrix, corresponding to the Coulomb interaction.

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Summary.

A formulation of quantum electrodynamics, without a supplementary condition, is given. Starting from the interaction representation, light waves are characterized by the 6-vector field satisfying the homogeneous Maxwell equations. In order to describe longitudinal interactions, an additional scalar field is introduced. Interactions with the electrons are defined by means of potentials given by these fields and related to a special time-like vector n_μ (or to a corresponding space-like surface). The scalar field variables can be eliminated by means of a canonical transformation which leads to a wave equation containing the transverse interaction energy and the Coulomb energy. In the Heisenberg representation, the potentials whose gauge is related to the special time-like vector n_μ do not satisfy the Lorentz condition. The field strength operators obey, however, the Maxwell equations. In calculating the S-matrix, the commutation rules of the potentials which depend on n_μ can be replaced by the simpler rules of the Fermi electrodynamics.

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